Rook Polynomials: A Straight-Forward Problem

Trine Mathematics Colloquium

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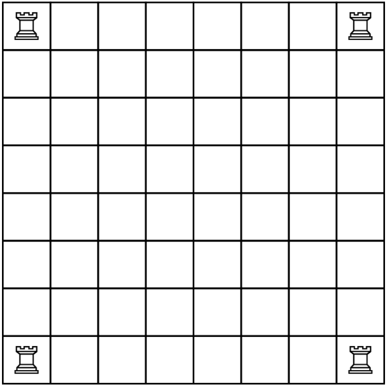
The Big Questions

- What is the rook problem?
- What are the different versions of the rook problem?
- What is a rook polynomial and what are they used for?
- How do different versions of the rook problem relate to one another?

Combinatorics

Combinatorics is the mathematical discipline of counting abstract objects.

The Rook Problem - Standard Chessboard



Pick a nonnegative integer k, then place k rooks on a chessboard so that none of the rooks can attack another.

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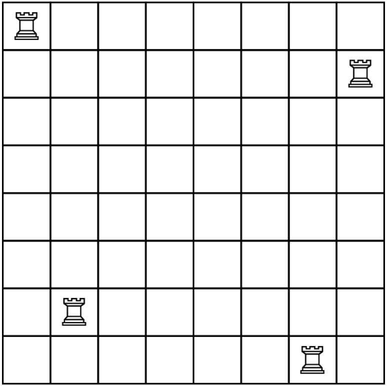

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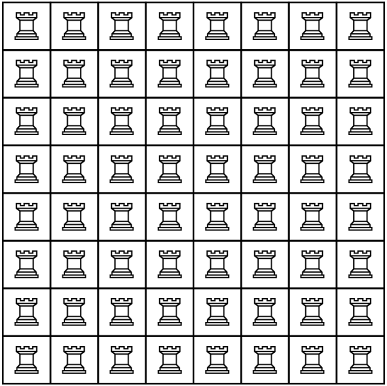
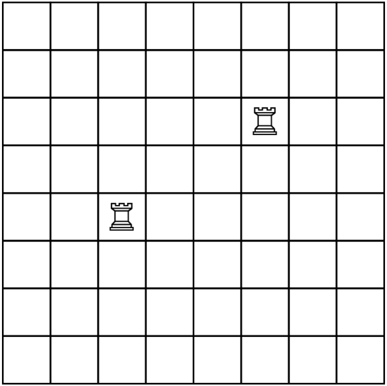


This might be impossible if k is too large.

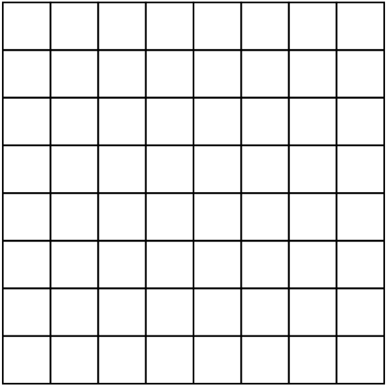


Here is a 2-rook solution of the standard chessboard. How many other solutions are there?

Let r_k denote the number of solutions to the k-rook problem on the 8×8 board. What's r_0?

k	0	1	2
r_k	?	?	?

How many ways can you place 0 rooks on a chessboard?



What's r_1?

k	0	1	2
r_k	1	?	?

There are 64 squares that a rook can occupy, so $r_1 = 64$.

What's r_2?

k	0	1	2
r_k	1	64	?

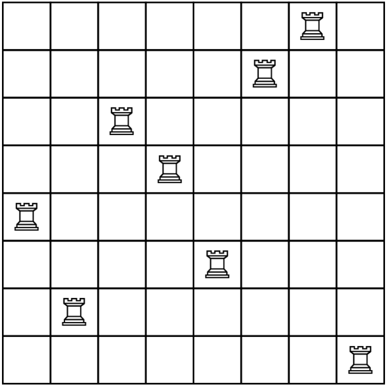
There are 64 squares that the first rook can occupy. This blocks out one whole row and one whole column. There are 49 spaces left that the second rook can occupy.

However, the order that you place these rooks down does not change the final arrangement. The correct answer is $r_k = 64 \cdot 49/2$, or 1568.

Continuing this reasoning, for $1 \leq k \leq 8$, we have

$$r_k = \frac{64 \cdot 49 \cdots (8-k+1)^2}{k!}.$$

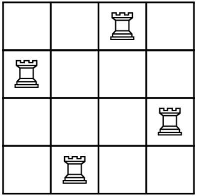
The numerator counts the number of ways to pick the squares, and the denominator makes sure we don't count the same solution multiple times.



Note that r_8 reduces to $8!$, which is $40\,320$.

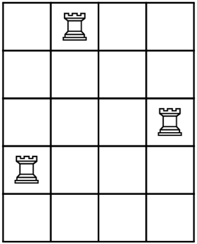
The Rook Problem - Other Square Boards

As a special case, let's look at k rooks on a $k \times k$ board. For these boards, $r_k = k!$.

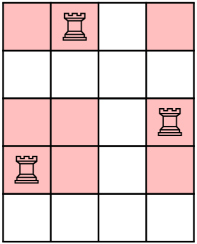


This is because there are k spaces that the first-row rook can occupy, times the $(k-1)$ spaces that the second-row rook can occupy, and so on. By the time we get to the last row, there is exactly one space left for the last rook.

The Rook Problem - Other Rectangular Boards



We're going to use our special case to count the number of solutions on $m \times n$ rectangular boards.



Suppose $0 \leq k \leq \min(m, n)$. We can place our rooks by first choosing k rows and k columns. This makes a $k \times k$ subboard. There will be $k!$ solutions for each different square sub-board. How many sub-boards are there?

Combinatorialists use the binomial coefficient $\binom{n}{k}$ to count the number of ways of choosing k objects from a set of n objects. It can be computed as

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}.$$

Then we just figured out that

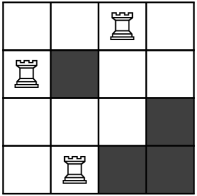
$$r_k = \binom{m}{k}\binom{n}{k}k!$$

for $m \times n$ boards.

Now let's look at the most complicated boards possible.

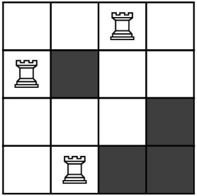
The Rook Problem - Boards with Holes

To study boards with holes in them, we will need more complicated sub-boards. We'll want to distinguish rook numbers that belong to different sub-boards, so let's label the rook numbers $r_k^{\mathcal{B}}$ to associate them to their board $\mathcal{B}$.

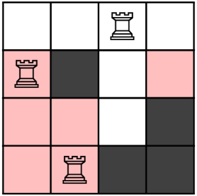


Careful, the $\mathcal{B}$ is not an exponent!

Pick a single space on the board. Each k-rook solution falls into one of two buckets: solutions with a rook in the chosen space, and solutions without a rook in the chosen space.

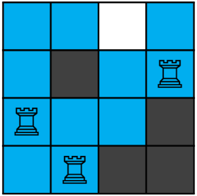


If there is a rook in the chosen space, highlight all the spaces that the rook cannot attack. We now have a $(k-1)$-rook solution on sub-board $\mathcal{B}_1$.



There are $r_{k-1}^{\mathcal{B}_1}$ of these solutions.

If there is not a rook in the chosen space, highlight all the but the chosen space. We now have a k-rook solution on sub-board $\mathcal{B}_2$.



There are $r_k^{\mathcal{B}_2}$ of these solutions.

This means that we can calculate the rook numbers for a large board in terms of the rook numbers of two smaller boards,

$$r_k^{\mathcal{B}} = r_{k-1}^{\mathcal{B}_1} + r_k^{\mathcal{B}_2}.$$

This is an example of a *recursive* problem. We'll use a *generating function* to finish up the job.

Generating Functions

Let $a(n)$ be a sequence defined for $n \geq 0$. The generating function of $a(n)$ is the series

$$\sum_{n=0}^{\infty} a(n)x^n = a(0) + a(1)x + a(2)x^2 + \cdots .$$

You can add, subtract and multiply these series according to the usual rules for algebra.

Generating Functions

For this problem, we already know that each board has a maximum capacity. So our generating functions are just polynomials,

$$\begin{aligned} R_{\mathcal{B}}(x) &= r_0^{\mathcal{B}} + r_1^{\mathcal{B}} x + \cdots + r_n^{\mathcal{B}} x^n \\ &= \sum_{k=0}^{n} r_k^{\mathcal{B}} x^k. \end{aligned}$$

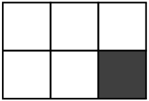
Therefore, every board has a *rook polynomial* which records all of its rook numbers.

In fact, the recursive formula for rook numbers translates to a recursive formula for the rook polynomials:

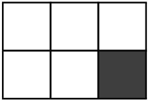
$$R_{\mathcal{B}}(x) = xR_{\mathcal{B}_1}(x) + R_{\mathcal{B}_2}(x).$$

One strategy for calculating the rook polynomials is to look for clever ways to break larger boards into pieces that we already understand.

Let's call this board $\mathcal{B}$.



If we choose the top-right corner to make our sub-boards, then we get two rectangular boards which we already know how to solve!

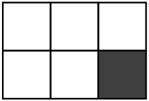


In this situation, $\mathcal{B}_1$ is a 2×1 rectangle. Its rook polynomial is $1 + 2x$.

Moving on, $\mathcal{B}_2$ is a 2×2 square. Its rook polynomial is $1 + 4x + x^2$.

Therefore, $R_{\mathcal{B}} = x(1 + 2x) + (1 + 4x + 2x^2) = 1 + 5x + 4x^2$.

Can you find all the solutions by hand?



$$R_{\mathcal{B}} = 1 + 5x + 4x^2.$$

Conclusion

Please join me again in the future when I discuss a more theoretical result between rook polynomials, the Rook Reciprocity Theorem.

Thank You!